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LETTER TO THE EDITOR

Integrable deformations of the Heisenberg model: prolongation structure technique

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Abstract. The integrable deformations of the one-dimensional classical Heisenberg model have been investigated by use of the prolongation theory of Wahlquist and Estabrook. The SO(3) invariant deformation and a higher-order isotropic deformation with nonlinearity in the deformation parameters are obtained. The Lax pairs for these integrable spin equations are presented. Finally it is pointed out that these spin equations are equivalent to the generalized nonlinear Schrödinger equations.

In reference [1], Mikhailov and Shabat investigated the first-order integrable deformations of the one-dimensional classical Heisenberg model by use of the equivalence between the Heisenberg spin equations and the integrable nonlinear Schrödinger equations. It is well known that all nonlinear evolution equations which are completely integrable exhibit non-Abelian prolongation structure [2, 3]. Recently, we also discussed the integrable deformations of the Heisenberg model by employing the prolongation structure technique [4]. The anisotropic deformations which are identical to those in reference [1] and the arbitrary N-order isotropic deformations were obtained. In this letter we shall construct the other integrable deformations using the prolongation theory.

First, we consider the following integrable spin equation

$$\boldsymbol{s}_t = \boldsymbol{s}_{xx} + \boldsymbol{E}(\boldsymbol{s}, \boldsymbol{s}_x) \qquad \boldsymbol{s}\cdot\boldsymbol{s} = 1 \qquad \boldsymbol{s}\cdot\boldsymbol{E} = 0 \tag{1}$$

where the deformation vector $\boldsymbol{E} = (E_1, E_2, E_3)$ is to be determined. Taking $s_{a,x}$ and $s_{a,xx}$ $(a = 1, 2, 3)$ as new independent variables. Then, (1) can be represented by the following set of 2-forms

$$\begin{aligned}
\alpha_1 &= \mathrm{d}s \wedge \mathrm{d}t - s_{1,x}\,\mathrm{d}x \wedge \mathrm{d}t \\
\alpha_{a+3} &= \mathrm{d}s_{a,x} \wedge \mathrm{d}t - s_{a,xx}\,\mathrm{d}x \wedge \mathrm{d}t \\
\alpha_{a+6} &= \mathrm{d}s_a \wedge \mathrm{d}x + \varepsilon_{abc}s_b\,ds_{c,x} \wedge \mathrm{d}t + E_a\,\mathrm{d}x \wedge \mathrm{d}t \\
\alpha_{a+9} &= \mathrm{d}s_{a,x} \wedge \mathrm{d}x + \varepsilon_{abc}s_b\,\mathrm{d}s_{c,xx} \wedge \mathrm{d}t + \varepsilon_{abc}s_{b,x}\,\mathrm{d}s_{c,x} \wedge \mathrm{d}t + E_{a,x}\,\mathrm{d}x \wedge \mathrm{d}t \\
\alpha_{13} &= s_a\,\mathrm{d}s_{a,xx} \wedge \mathrm{d}t + 3s_{a,x}\,\mathrm{d}s_{a,x} \wedge \mathrm{d}t \equiv 0
\end{aligned} \tag{2}$$

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where ε_{abc} is the totally antisymmetric tensor and the summation convention for repeated indices has been used. It is worthwhile noting that 2-forms α_{a+9} come from differentiating (1) with respect to the space variable x. It is not difficult to prove that the ideal, I, of 2-forms (2) is closed, i.e. $\mathrm{d}\alpha_i = f_{ij} \wedge \alpha_j$; f_{ij} is some set of 1-forms. To obtain prolongation structure [2], we introduce a set of 1-forms

$$\omega_k = \mathrm{d}y^k + F^k\,\mathrm{d}x + G^k\,\mathrm{d}t \qquad k = 1, 2, \ldots, n \tag{3}$$

where y^k are the prolongation variables, F^k and G^k are functions of s_a, $s_{a,x}$, $s_{a,xx}$ and y^k. Demanding the extended ideal $\tilde{I} = \{\alpha_i, \omega_k\}$ to be closed (i.e. $\mathrm{d}\omega_k = g_i^k\alpha_i + n_l^k \wedge \omega_l$, g_i^k and n_l^k are 0-forms and 1-forms, respectively) leads to the integrability conditions for (1)

$$F^k_{s_{a,xx}} = 0$$

$$G^k_{s_{a,xx}} = \varepsilon_{abc} F^k_{s_{b,x}} s_c \tag{4}$$

$$s_{a,x} G^k_{s_a} + s_{a,xx} G^k_{s_{a,x}} + \varepsilon_{abc} s_{a,xx}(s_b F^k_{s_c} + s_{b,x} F^k_{s_{c,x}}) - E_a F^k_{s_a} - E_{a,x} F^k_{s_{a,x}} - [F, G]^k = 0$$

where $[F, G]^k = F^l G^k_{y_l} - G^l F^k_{y^l}$. Because the anisotropic integrable deformations were obtained [4], we only give here the isotropic deformation. Equations (4) have the solutions

$$F = 2\alpha\lambda^2 s_a X_a + 2\alpha\lambda s_{a,x} X_a \tag{5a}$$

$$\begin{aligned} G = {} & 2\alpha\lambda\varepsilon_{abc} s_a s_{b,xx} X_c + 2\alpha\lambda^2 \varepsilon_{abc} s_a s_{b,x} X_c \\ & + (2\alpha\beta\lambda - 4\alpha^2\lambda^3 + 2\alpha^2\lambda \boldsymbol{s}_x \cdot \boldsymbol{s}_x) s_{a,x} X_a \\ & + (2\alpha^2\lambda^2 \boldsymbol{s}_x \cdot \boldsymbol{s}_x + 2\alpha\beta\lambda^2 - 4\alpha^2\lambda^4) s_a X_a \end{aligned} \tag{5b}$$

$$\boldsymbol{E} = \alpha \boldsymbol{s}_x \cdot \boldsymbol{s}_x \boldsymbol{s}_x + \beta \boldsymbol{s}_x \tag{5c}$$

where α and β are the deformation constants, λ is an arbitrary constant as in the spectral parameter below, X_a depend only on the prolongation variables y^k and constitute the su(2) Lie algebra, i.e. $[X_a, X_b] = \varepsilon_{abc} X_c$.

The Lax pair for (1) is

$$U = F|_{X_a} = -\frac{\mathrm{i}}{2}\sigma_a = -\mathrm{i}\alpha\lambda^2 S - \mathrm{i}\alpha\lambda S_x$$

$$\begin{aligned} V = G|_{X_a} = -\frac{\mathrm{i}}{2}\sigma_a = {} & -\tfrac{1}{2}\alpha\lambda[S, S_{xx}] - \tfrac{1}{2}\alpha\lambda^2[S, S_x] \\ & -\mathrm{i}(\alpha\beta\lambda - 2\alpha^2\lambda^3 + \alpha^2\lambda S_x^2) S_x - \mathrm{i}(\alpha\beta\lambda^2 - 2\alpha^2\lambda^4 + \alpha^2\lambda^2 S_x^2) S \end{aligned} \tag{6}$$

where σ_a are Pauli matrices and $S = s_a\sigma_a$. The deformation (5c) is nothing but the SO(3) invariant integrable deformation of the Heisenberg model [1].

Secondly, we consider the higher-order integrable spin equattion

$$\begin{aligned} & \boldsymbol{s}_t = \boldsymbol{s} \times \boldsymbol{s}_{xx} + \boldsymbol{E}(\boldsymbol{s}, \boldsymbol{s}_x, \boldsymbol{s}_{xx}) + \alpha \boldsymbol{s}_{xxx} + 3\alpha \boldsymbol{s}_x \cdot \boldsymbol{s}_{xx} \boldsymbol{s} \\ & \boldsymbol{s} \cdot \boldsymbol{s} = 1 \qquad \boldsymbol{s} \cdot \boldsymbol{E} = 0 \end{aligned} \tag{7}$$

where α is the deformation parameter. Repeating the same procedures above, we obtain the integrability conditions for (7)

$$F^k_{s_{a,xx}} = F^k_{s_{a,xxx}} = 0$$

$$G^k_{s_{a,xxx}} = \alpha F^k_{s_{a,x}}$$

$$\begin{aligned} & s_{a,x} G^k_{s_a} + s_{a,xx} G^k_{s_{a,x}} + s_{a,xxx} G^k_{s_{a,xx}} + \varepsilon_{abc} s_{a,xx} (s_b F^k_{s_c} + s_{b,x} F^k_{s_{c,x}}) \\ & \quad + \varepsilon_{abc} s_{a,xxx} s_b F^k_{s_{c,x}} - \alpha s_{a,xxx} F^k_{s_a} - E_a F^k_{s_a} - E_{a,x} F^k_{s_{a,x}} - 3\alpha s_{xx} \cdot s_{xx} s_a F^k_{s_{a,x}} \\ & \quad - 3\alpha s_x \cdot s_{xxx} s_a F^k_{s_{a,x}} - 3\alpha s_x \cdot s_{xx} (s_a F^k_{s_a} + s_{a,x} F^k_{s_{a,x}}) - [F, G]^k = 0. \end{aligned} \qquad (8)$$

We assume that (8) have the following solutions

$$F = A s_a X_a + B s_{a,x} X_a$$

$$\begin{aligned} G = {} & \alpha\beta s_{a,xxx} X_a + C \varepsilon_{abc} s_{a,x} s_{b,xx} X_c + D \varepsilon_{abc} s_a s_{b,xx} X_c \\ & + H s_{a,xx} X_a + I \varepsilon_{abc} s_a s_{b,x} X_c + J s_{a,x} X_a + K s_a X_a \end{aligned} \qquad (9b)$$

$$E = \xi s_x + \eta s \times s_x + \nu s \times s_{xx} + \theta s_{xx} + \theta s_x \cdot s_x s \qquad (9c)$$

where A and B, which contain the spectral parameter λ, are constants, C, D, H, I, J and K are functions of s_a, $s_{a,xx}$ and λ, ξ, η, ν and θ are functions of s_a, $s_{a,x}$ and $s_{a,xx}$. Substituting (9) into (8), we finally obtain

$$\begin{aligned} & A = -\frac{3\alpha}{2\beta} B^2(\lambda) \qquad \xi = \tfrac{3}{2}\alpha s_x \cdot s_x - \frac{\beta}{3\alpha} s_x \cdot s_x - \frac{\beta^2}{6\alpha}(s_x \cdot s_x)^2 + \gamma \qquad \nu = \beta s_x \cdot s_x \\ & \eta = \theta = 0 \qquad C = \alpha B^2 \qquad D = B + \alpha AB + B\nu \qquad H = \alpha A \\ & I = A + \alpha A^2 + A\nu \qquad J = B\xi - AB - \alpha A^2 B - AB\nu - \alpha B^3 s_x \cdot s_x \\ & K = 3\alpha B^3 s_x \cdot s_{xx} + AB\nu_x - \alpha AB^2 s_x \cdot s_x - A^2 - \alpha A^3 - A^2\nu + 3\alpha B s_x \cdot s_{xx} + A\xi \end{aligned} \qquad (10)$$

We note that the deformation (9c) is nonlinear in terms of the deformation parameters α and β. When $\beta = 0$, (7) becomes that equation discussed by Papanicolaou [5]; when $\beta = \varepsilon\alpha$ and $\alpha = 0$, (7) becomes the SO(3) invariant deformed Heisenberg spin equation (1).

Finally, we show that (7) is equivalent to a generalized nonlinear Schrödinger equation which is also completely integrable.

Following Lakshmanan [6], we map (7) on a moving helical space curve described by the orthogonal trihedral e_a which satisfies the Serret–Frenet equations

$$e_{1x} = k e_2 \qquad e_{2x} = -k e_1 + \tau e_3 \qquad e_{3x} = -\tau e_2 \qquad (11)$$

where the curvature is given by $k = (e_{1x} \cdot e_{1x})^{1/2}$ and the torsion is given by $\tau = k^{-2} e_1 \cdot (e_{1x} \times e_{1xx})$.

Taking $e_1 = s$, then, (7) can be rewritten as

$$\begin{aligned} e_{1t} = {} & \left(-k\tau + \alpha k_{xx} - \alpha k \tau^2 + \tfrac{1}{2}\sigma k^3 - \frac{\beta}{3\alpha} k^3 - \frac{\beta^2}{6\alpha} k^5 - \beta k^3 \tau \right) e_2 \\ & + (k_x + 2\alpha k_x \tau + \alpha k \tau_x + \beta k^2 k_x) e_3. \end{aligned} \qquad (12)$$

Using (11) and (12) and the constraints $e_{\alpha tx}=e_{\alpha xt}$, we obtain the following nonlinear evolution equations for the curvature k and the torsion τ of the space curve

$$k_t=-2k_x\tau-k\tau_x+\tfrac{3}{2}\alpha k^2k_x-\frac{\beta}{\alpha}k^2k_x-\frac{5\beta^2}{6\alpha}\kappa^4k_x-4\beta k^2k_x\tau-\beta k^3\tau_x+\alpha k_{xxx}$$
$$-3\alpha k_x\tau^2-3\alpha k\tau\tau_x$$
$$\tau_t=\Big(-\tau^2+\tfrac{1}{2}k^2+\tfrac{1}{4}\beta k^4+\tfrac{3}{2}\alpha k^2\tau-\frac{\beta}{3\alpha}k^2\tau-\frac{\beta^2}{6\alpha}k^4\tau-\beta k^2\tau^2+3\alpha k^{-1}k_{xx}\tau-\alpha\tau^3$$
$$+k^{-1}k_{xx}+2\beta k_x^2+\beta kk_{xx}+3\alpha k^{-1}k_x\tau_x+\alpha\tau_{xx}\Big)_x. \tag{13}$$

Making the complex transformation [6] we have

$$\psi(x,t)=k(x,t)\exp\left[\mathrm{i}\int_{-\infty}^{x}\tau(y,t)\,\mathrm{d}y\right]. \tag{14}$$

Equations (13) then become the generalized nonlinear Schrödinger equation

$$\mathrm{i}\psi_t+\psi_{xx}+\tfrac{1}{2}|\psi|^2\psi+\tfrac{1}{4}\beta|\psi|^4\psi+\frac{\mathrm{i}}{6\alpha}\beta^2(|\psi|^4\psi)_x$$
$$+\beta(|\psi|^2\psi_x)_x+\frac{\mathrm{i}}{3\alpha}\beta(|\psi|^2\psi)_x-\tfrac{3}{2}\mathrm{i}\alpha|\psi|^2\psi_x-\mathrm{i}\alpha\psi_{xxx}=0 \tag{15}$$

which is equivalent to (7). When $\beta=0$, the equivalence between the Papanicolaou equation and the Hirota equation is obtained [5]. When $\beta=\varepsilon\alpha$ and $\alpha=0$, the equivalence between the SO(3) invariant deformed Heisenberg spin equation and that which is a mixture of the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger equation is obtained [7].

The Lax pair for (15) is

$$U=\frac{3\mathrm{i}\alpha}{4\beta}(1+\lambda^2)\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\tfrac{1}{2}\lambda\begin{pmatrix}0&\psi\\\psi^*&0\end{pmatrix}$$
$$V=A\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\tfrac{1}{2}\lambda\begin{pmatrix}0&B\\B^*&0\end{pmatrix} \tag{16}$$

where

$$A=\tfrac{1}{4}\lambda^2\alpha(\psi_x^*\psi-\psi^*\psi_x)-\frac{\mathrm{i}}{8}\lambda^2\beta|\psi|^4-\frac{\mathrm{i}}{4}\lambda^2|\psi|^2-\frac{3\mathrm{i}\lambda^2\alpha^2}{8\beta}(1+\lambda^2)|\psi|^2$$
$$-\frac{9\mathrm{i}\alpha^2}{8\beta^2}(1+\lambda^2)^2-\frac{27\mathrm{i}\alpha^4}{16\beta^3}(1+\lambda^2)^3$$
$$B=\alpha\psi_{xx}-\frac{\beta}{3\alpha}|\psi|^2\psi+\mathrm{i}\beta|\psi|^2\psi_x-\frac{\beta^2}{6\alpha}|\psi|^4\psi+\mathrm{i}\psi_x+\frac{3\mathrm{i}\alpha^2}{2\beta}(1+\lambda^2)\psi_x$$
$$-\frac{9\alpha^3}{4\beta^2}(1+\lambda^2)^2\psi-\frac{3\alpha}{2\beta}(1+\lambda^2)\psi-\tfrac{1}{2}\alpha\lambda^2|\psi|\psi. \tag{17}$$

In conclusion, we have investigated the integrable deformations of the Heisenberg model by use of prolongation theory. It is interesting that the deformation (7) is nonlinear in terms of the deformation parameters α and β. From (8), we have not obtained the higher-order anisotropic integrable deformations of the Heisenberg model [8, 9].

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